Module P2.2 Projectile motion

Exit module

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1 Opening items

1.1 Module introduction

Any object that is launched into unpowered flight close to the Earth's surface is called a *projectile*. Examples include artillery shells, shot putts, high jumpers and balls in games such as tennis, football and cricket. To a good approximation (ignoring any sideways swerve) many projectiles can be modelled as particles moving along curved paths in a vertical plane. The motion is therefore two-dimensional and the following questions can be asked:

What is the horizontal range of the projectile and what is the maximum height that it attains?

How long does the flight take, and what is the shape of the flight path?

How do the velocity and acceleration of the projectile vary during its flight?

This module explains how questions such as these can be answered. In doing so it also provides an introduction to the general analysis of two-dimensional motion in terms of *vectors* and their (*scalar*) *components*. In particular, Section 2 explains how vectors can be used to describe quantities such as the *position*, *displacement*, *velocity* and *acceleration* of a projectile, and how vectors of a similar type can be added together and algebraically manipulated.

Section 3 investigates the equations that describe the motion of a projectile, stressing the fact that the horizontal and vertical motions of a projectile are essentially independent, apart from having the same duration. The equations that emerge from this investigation are used to deduce a number of general features of projectile motion, including the shape of the projectile's *trajectory* and the condition for achieving the maximum horizontal *range*. Finally, in Section 4, the techniques developed earlier are used to solve a variety of two-dimensional projectile problems, and their extension to three-dimensional problems involving arbitrary uniform acceleration is briefly mentioned.

The following problem will give you an idea of the sort of question you will be able to answer by the end of this module. The solution is given in Subsection 4.3.

> An aircraft flies at a height of 2000 m with a constant velocity of $150\,\mathrm{m\,s^{-1}}$ in a straight horizontal line. As it passes vertically over a gun a shell is fired from the gun. Find the *minimum* muzzle speed, u, of the shell, and the angle ϕ, from the horizontal, at which the shell should be fired in order for it to hit the plane. Take the magnitude of the acceleration due to gravity g as $9.81\,\mathrm{m\,s^{-2}}$ ☞ and ignore air resistance.

Study comment Having read the introduction you may feel that you are already familiar with the material covered by this module and that you do not need to study it. If so, try the *Fast track questions* given in Subsection 1.2. If not, proceed directly to *Ready to study?* in Subsection 1.3.

1.2 Fast track questions

Study comment Can you answer the following *Fast track questions*?. If you answer the questions successfully you need only glance through the module before looking at the *Module summary* (Subsection 5.1) and the *Achievements* listed in Subsection 5.2. If you are sure that you can meet each of these achievements, try the *Exit test* in Subsection 5.3. If you have difficulty with only one or two of the questions you should follow the guidance given in the answers and read the relevant parts of the module. However, *if you have difficulty with more than two of the Exit questions you are strongly advised to study the whole module*.

Question F1

Miss Magnificent, the human cannonball, is fired from her cannon with a muzzle speed of $18\,\mathrm{m\,s^{-1}}$ at an angle of 35° to the horizontal. If she is to land at the same level from which she took off, how far away from the cannon should the net in which she wishes to land be placed? You should ignore the effect of air resistance and treat Miss Magnificent as a point particle.

Question F2

A particle moves in the (x, y) plane from point 1, with position (3 m, 2 m) to point 2, with position (7 m, −1 m).

(a) What are the x- and y-components of the particle's displacement from point 1 to point 2?

(b) What is the magnitude of this displacement?

(c) What is the angle between this displacement and the x-axis?

Question F3

A shell is fired with a velocity of $200\,\mathrm{m\,s^{-1}}$ at an angle of 30° above the horizontal. Find the time taken for the shell to reach its maximum height and the magnitude and direction of its velocity after 16 s. Take the magnitude of the acceleration due to gravity to be $g = 9.81\,\mathrm{m\,s^{-2}}$ and ignore the effect of air resistance.

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Study comment Having seen the *Fast track questions* you may feel that it would be wiser to follow the normal route through the module and to proceed directly to *Ready to study?* in Subsection 1.3.

Alternatively, you may still be sufficiently comfortable with the material covered by the module to proceed directly to the *Closing items*.

1.3 Ready to study?

Study comment To begin to study of this module you will need to be familiar with the following terms or topics: *Cartesian coordinate system*, *cosine* (as in cos θ), *gradient of a line*, *graph*, *position coordinates*, *Pythagoras's theorem*, *quadratic equation*, *SI units*, *sine* (as in sin θ) and *tangent* (as in tan θ). It is also assumed that you have some familiarity with concepts such as *position*, *speed*, *velocity* and *acceleration* in the context of *one-dimensional linear motion* and that you have previously encountered the calculus notation dx/dt used to indicate the (instantaneous) *rate of change* of x with respect to t. However, proficiency in the use of calculus is *not* required for the study of this module. If you are uncertain about any of these items you can review them now by reference to the *Glossary*, which will also indicate where in *FLAP* they are developed. The following *Ready to study questions* will enable you to check whether you need to review some of the topics before embarking on this module.

Question R1

A right-angled triangle has sides of length a, b and c. If the longest side is of length c and if the angle between that side and side b is θ, write down Pythagoras's theorem as it applies to this particular triangle and express the lengths a and b in terms of c and θ.

Question R2

An object moving along the x-axis of a Cartesian coordinate system does so in such a way that its position coordinate x changes from $x = -1.0\,\text{m}$ to $x = 8.0\,\text{m}$ in a time interval of 3.0 s. What is the average velocity $\langle v_x \rangle$ of the object over the three-second interval? In what way would your answer have been different if the initial and final values of x had been interchanged?

Question R3

A cyclist is travelling due north along a straight road at $10\,\text{m}\,\text{s}^{-1}$ when the brakes are applied. If the brakes can cause the speed to reduce at a constant rate of $3\,\text{m}\,\text{s}^{-2}$, how long will it take to stop the bicycle and how far will it have travelled in that time?

2 Describing the motion of a projectile

Any object that is launched into unpowered flight near the Earth's surface is called a **projectile**. A rock thrown from a hand or an arrow released from a bow would be typical examples. The motion of a projectile has been of interest since ancient times, but describing the motion accurately and explaining its cause has created a great deal of difficulty. In the 4th century BC the Greek philosopher Aristotle (384–322BC) said that when an object is thrown it will carry on in a straight line until it runs out of '*force*', after which it will fall down. Figure 1 illustrates his idea, which we will soon see to be incorrect.

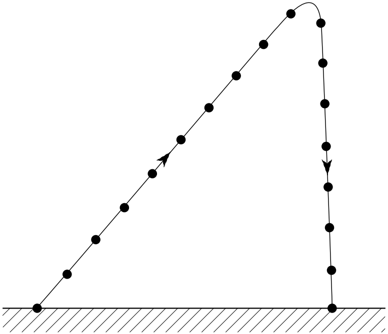

Figure 1 Aristotle's idea of projectile motion. The dots represent successive positions of the projectile at equally separated moments of time.

In medieval times the exact paths of cannonballs and arrows were of great interest, and it became necessary to think about them more precisely.

In the 16th century the Italian scientist Galileo Galilei (1564–1642) produced an accurate analysis of the flight path, or **trajectory**, of a projectile, this is illustrated schematically in Figure 2.

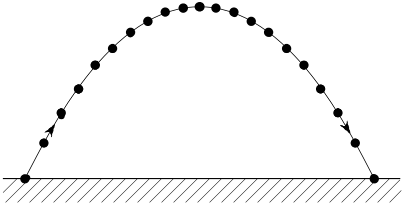

Figure 2 Galileo's idea of projectile motion. Note how the dots representing successive positions of the projectile bunch together near the top of the trajectory.

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In this module we will be concerned only with projectiles that may be treated as **particles** (i.e. bodies with negligible size and no internal structure, that may be regarded as occupying a single point in space at any particular time). Also we will assume that during flight, projectiles move under the influence of gravity alone; in other words we will ignore *air resistance* and other such effects. These restrictions may seem rather limiting, but for many purposes the motion of quite large bodies, such as footballs and people, can often be modelled quite successfully on this basis.

Of course, there are many problems in which these simplifications would be misleading. Real projectiles have a finite size and this allows air resistance to influence their motion. When air resistance is large it cannot be ignored, for example, badminton players will recognize the trajectory of a shuttlecock as being nearer to that shown in Figure 1 than Figure 2. Also, in many ball games, such as cricket and tennis, it is possible to spin the ball. The effect of this spin on the flow of air around the ball produces a sideways force so that the ball 'swerves'. These phenomena are beyond the scope of this module, though it is not too difficult to extend the ideas that are introduced here in order to accommodate them.

2.1 Position and displacement

Study comment This subsection is mainly concerned with vector notation and deliberately parallels the development of *vectors* presented in the maths strand of *FLAP*. If you have already studied vectors there or elsewhere you should aim to finish this subsection very quickly, pausing only to make sure that you are familiar with the notation that will be used in this module and that you are able to answer Questions T1 and T2. If you are unfamiliar with vectors and you feel you need a more detailed treatment than that presented here you should consult the *Glossary* which will refer you to the appropriate modules in the maths strand.

Position

A projectile represented by a point particle moving under the influence of gravity has a trajectory that is confined to a single vertical plane. Such a projectile is said to exhibit **two-dimensional** motion, since its trajectory may be adequately described using just two independent *coordinate axes*. In this module we will use two mutually-perpendicular *Cartesian coordinate axes* for this purpose: a horizontal axis which will usually be labelled x, and a vertical axis which will usually be labelled y. These axes meet at a point called the *origin* from which we can measure the *position coordinates* of any point in the plane of the axes (i.e. the specific values of x and y that determine the location of that point in the plane). So, provided we choose the directions of the x- and y-axes appropriately, we can suppose that the motion of any projectile we wish to consider is confined to the (x, y) plane and may be described in terms of the changing values of the projectile's x- and y-coordinates.

Figure 3 uses a two-dimensional Cartesian coordinate system of the sort just described to show various points along the trajectory of a projectile launched from the origin (the point marked O). As usual, the dots represent successive positions of the projectile at equally separated moments of time. The location of any point on the trajectory, such as the point marked A, can be described in terms of its position coordinates. In this particular case we can call them $x_{\rm A}$ and $y_{\rm A}$ since they relate to the point A, and we can adopt the convention of writing them as an **ordered pair** $(x_{\rm A}, y_{\rm A})$, in which the first value is always the *x*-coordinate and the second the *y*-coordinate.

✦ What are the position coordinates $(x_{\rm A}, y_{\rm A})$ of point A in Figure 3?

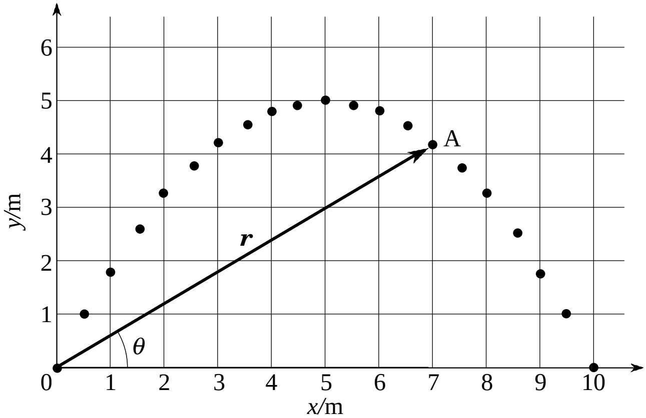

Figure 3 The arrow drawn from the origin to a point such as A represents the *position vector* ***r*** of that point.

Figure 3 also indicates an alternative way of specifying the location of a point in the (x, y) plane. An arrow of given length and given orientation, with its tail at the origin, will inevitably have its tip at some specific point. Such an arrow is called the **position vector** of the point and is generally denoted by the bold symbol $\boldsymbol{r}$. The bold symbol is used to distinguish the position vector (which has both magnitude *and* direction) from its magnitude r which is the **distance** of the point from the origin of coordinates. A distance r doesn't point in any particular direction and does not specify a point; a position vector $\boldsymbol{r}$ has direction and does specify a point.

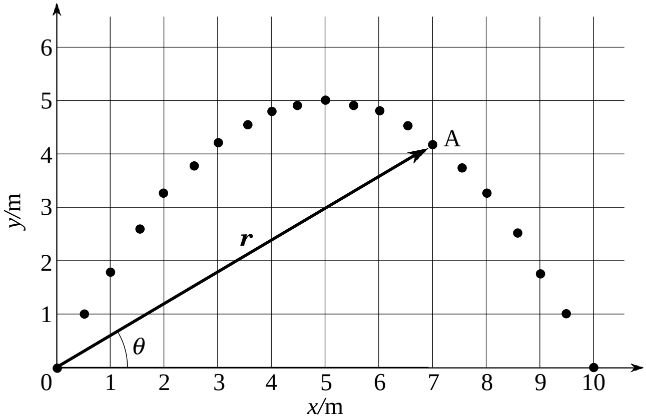

Figure 3 The arrow drawn from the origin to a point such as A represents the *position vector* $\boldsymbol{r}$ of that point.

Note that in order to specify the position vector $\boldsymbol{r}$ of a particular point we must clearly define two things:

1 the **magnitude** of $\boldsymbol{r}$, which is the distance r from the origin to the point in question;

2 the **direction** of $\boldsymbol{r}$, which in two dimensions can be described by the angle θ measured in an anticlockwise direction from the positive x-axis to $\boldsymbol{r}$.

✦ Using a ruler and a protractor determine the magnitude and direction of the position vector $\boldsymbol{r}$ of point A. ☞

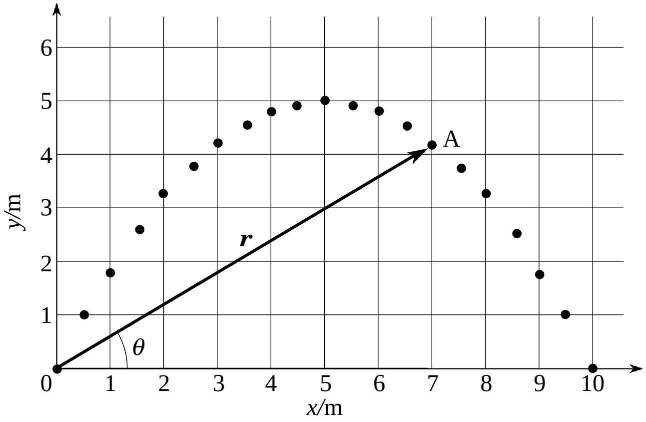

Figure 3 The arrow drawn from the origin to a point such as A represents the *position vector* $\boldsymbol{r}$ of that point.

Aside As has already been stressed, position vectors and their magnitudes may both be represented by the same letter, but the vector is distinguished by the use of a bold typeface. This is fine in print but not of much help in work that you have to write by hand. In order to show that a handwritten letter is supposed to represent a vector you should generally put a wavy underline beneath it as in $\underset{\sim}{a}$. This sort of underline is used to indicate to a printer that whatever has been underlined should be set in bold type.

So, we now have two different ways of specifying the location of any point in the (x, y) plane. We can either give the coordinates of the point (x, y) or we can specify the point's position vector $\boldsymbol{r}$ by giving its magnitude and direction. There is clearly an intimate relationship between these two descriptions. Indeed, it is natural to use the ordered pair of position coordinates (x, y) as a way of identifying the position vector $\boldsymbol{r}$ by writing

$$\boldsymbol{r} = (x, y) \tag{1}$$

When represented in this way we say that x and y are the **components** of the position vector $\boldsymbol{r}$. Thus, we may say that the point A in Figure 3 has the position vector, $\boldsymbol{r}$ = (7.0 m, 4.2 m) or, equivalently, we may say that the x-component of $\boldsymbol{r}$ is 7.0 m, and that the y-component of $\boldsymbol{r}$ is 4.2 m.

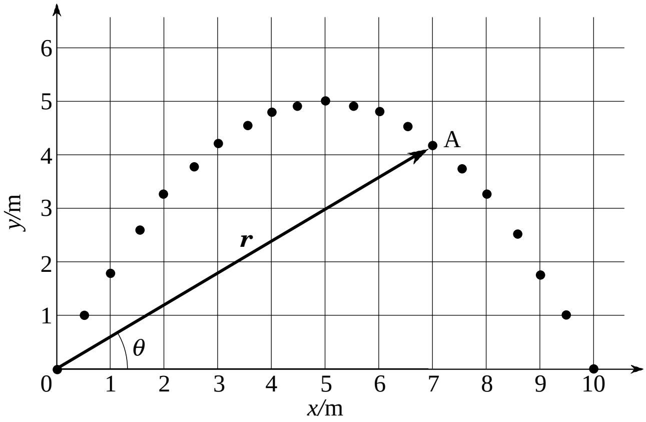

Figure 3 The arrow drawn from the origin to a point such as A represents the *position vector* $\boldsymbol{r}$ of that point.

For a two-dimensional position vector, the magnitude r is related to the components x and y by *Pythagoras's theorem*:

$$r = \sqrt{x^2 + y^2} \qquad (2)$$

where $\sqrt{x^2 + y^2}$ indicates the *positive* square root of $x^2 + y^2$. Note that this relation ensures that the magnitude of the position vector will be a *positive* quantity as any length must be. It is often useful to emphasize this by using the symbol $|\boldsymbol{r}|$ to represent the magnitude of $\boldsymbol{r}$. You might forget that r represents a positive quantity, but you are unlikely to forget that $|\boldsymbol{r}|$ must be positive.

The direction of a position vector may also be related to its components. For the two-dimensional position vector $\boldsymbol{r} = (x, y)$, the direction of which is specified by the angle θ, the relationship takes the form

$$\tan\theta = \frac{y}{x} \qquad (3)$$

It also follows from basic trigonometry that the horizontal and vertical components of a position vector $\boldsymbol{r}$ are

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta \qquad (4)$$

so $\boldsymbol{r} = (r\cos\theta, r\sin\theta)$ (5)

Question T1

Figure 4 shows a point P and its position vector, $\boldsymbol{r}$. Write down the position coordinates of the point, P, and use those coordinates to determine the magnitude and direction of the position vector $\boldsymbol{r}$. ❑

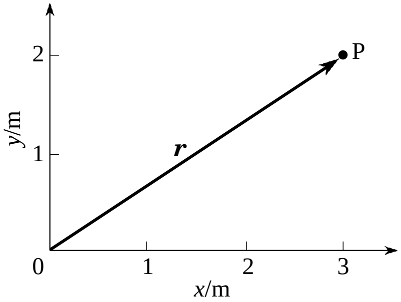

Figure 4 A point P with its position vector, $\boldsymbol{r}$.

Position vectors are not the only quantities with magnitude and direction that are of interest to us in the study of projectile motion, or indeed in physics generally. In fact, any quantity that requires both a magnitude and a direction for its complete specification is called a **vector**, and common examples that you will meet in this module include *velocity* and *acceleration*. As you will see later, any vector may be expressed in terms of its *components* though in most cases the components will not have such a simple interpretation as the components of a position vector.

Many physical quantities, such as mass, length, time, speed and temperature, are specified by a magnitude alone. Such quantities are collectively called **scalars**. Although scalars are quite distinct from vectors it should be noted that the *components* of a vector are actually *scalar* quantities (they are sometimes formally referred to as *scalar components* for this reason).

Thus, although it would not make sense to equate a vector (which involves direction) with a scalar (which does not), it *is* possible to identify a vector with an ordered arrangement of scalars such as (7.0 m, 4.2 m) since it is the agreed ordering and the predefined coordinate system that supplies the directional information.

> A vector is a quantity that is characterized by a magnitude *and* a direction. By convention, bold typeface symbols such as $\boldsymbol{r}$ are used to represent vector quantities. The magnitude of such a vector is represented by $|\boldsymbol{r}|$ or r and is a non-negative scalar quantity. In diagrams, vectors are represented by arrows or directed line segments. In handwritten work vectors are distinguished by a wavy underline ($\underset{\sim}{a}$) and the magnitude of a vector is denoted $|\underset{\sim}{a}|$.

Displacement

In many projectile problems a vector quantity that is of particular interest is **displacement** which can be used to describe a change or a difference in position. The distinction between the position and displacement vectors can be seen from Figure 5 in which O is the origin of a Cartesian coordinate system from which the changing position of a projectile can be measured. During part of its motion the projectile moves from P_1 (with position vector $\boldsymbol{r}_1 = (x_1, y_1)$) to P_2 (with position vector $\boldsymbol{r}_2 = (x_2, y_2)$). The corresponding change (or difference) in the projectile's position can be represented by the arrow marked $\boldsymbol{s}$ in Figure 5. This is certainly a vector quantity since it has both magnitude and direction, but it cannot be a position vector because its tail is not at the origin. It is in fact the *displacement* from P_1 to P_2.

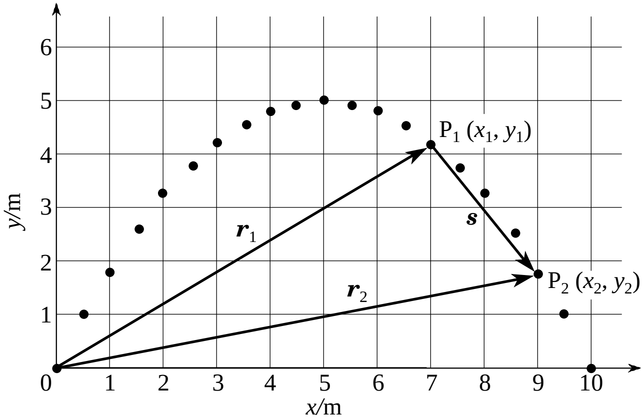

Figure 5 Displacement and position vectors.

Like all vectors, the displacement $\boldsymbol{s}$ from P_1 to P_2 can be expressed in terms of its components. These are generally denoted by s_x and s_y; so we can write $\boldsymbol{s} = (s_x, s_y)$, where s_x and s_y, respectively, represent the differences in the x- and y-coordinates of P_1 and P_2:

$$s_x = (x\text{-coordinate of } P_2) - (x\text{-coordinate of } P_1)$$

$$= x_2 - x_1$$

$$s_y = (y\text{-coordinate of } P_2) - (y\text{-coordinate of } P_1)$$

$$= y_2 - y_1$$

Thus, $$\boldsymbol{s} = (s_x, s_y) = (x_2 - x_1, y_2 - y_1) \qquad (6)$$

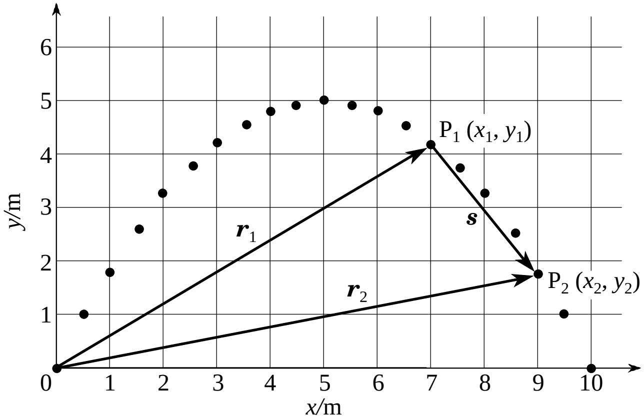

Figure 5 Displacement and position vectors.

✥

What are the components of the particular displacement shown in Figure 5? Use your answers to write the vector as an ordered pair of components.

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The great advantage of using displacements rather than position vectors when describing projectile motion is that the displacement from one point to another does not depend in any way on where the origin of the coordinate system is located. For instance, even if we moved the origin of coordinates in Figure 5 from the origin to some other point, it would still remain true that the displacement from P_1 to P_2 is $\boldsymbol{s} = (2.0\,\text{m}, -2.4\,\text{m})$; choosing a new origin would change the position coordinates and position vectors of P_1 and P_2, but it would not change the *differences* that determine s_x and s_y. ☞ Why is this an advantage? In projectile problems we are usually interested in how far the projectile has travelled (horizontally or vertically) from its launch point. If we describe the motion in terms of displacements from the launch point rather than positions relative to the origin we will have a description (in terms of displacements) that will be equally valid whether or not the launch point happens to be at the origin.

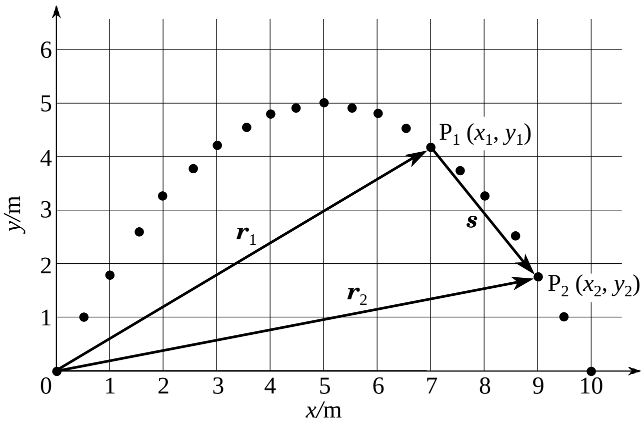

Figure 5 Displacement and position vectors.

In Figure 5 the launch point *was* at the origin, but that won't always be the case.

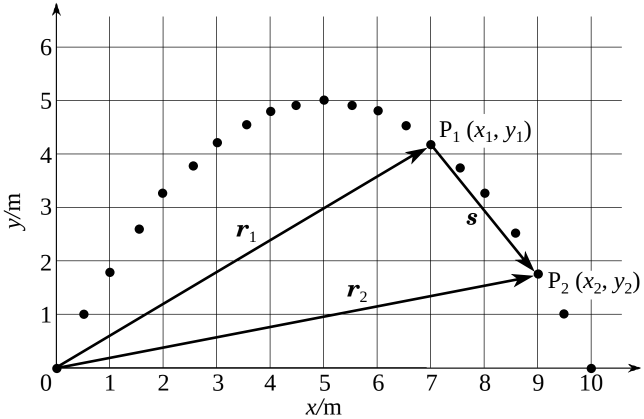

Figure 5 Displacement and position vectors.

Although we have gone to some lengths to stress the differences between position vectors and displacements it is important to realize there are also deep similarities. A displacement vector provides us with a way of describing the location of one point (such as P_2) with respect to some arbitrarily chosen reference point (such as P_1). A position vector provides us with a way of describing the location of one point (such as P_2) with respect to a very particular reference point — the origin O. Clearly, if we were to choose O as the arbitrary reference point from which we measured displacements then the displacement of a point such as P_2 would be equal to its position vector. To this extent position vectors are simply a special class of displacements; *position vectors are displacements from the origin.*

2.2 Vector algebra

It is a general property of displacements that they can be added together, though the word 'added' has to be interpreted in a rather special way. For example, the projectile launched from point O in Figure 5 will at some stage arrive at point P_1, where its displacement from O is $\boldsymbol{r}_1$, ☞ and some time later, after undergoing a further displacement $\boldsymbol{s}$ from P_1, it will arrive at P_2 where its displacement from O is $\boldsymbol{r}_2$. Consequently it makes sense to say that the result of adding the displacement $\boldsymbol{s}$ to the displacement $\boldsymbol{r}_1$, is the displacement $\boldsymbol{r}_2$. More formally we say that $\boldsymbol{r}_2$ is the **vector sum** or **resultant** of $\boldsymbol{r}_1$ and $\boldsymbol{s}$ and we write

$$\boldsymbol{r}_2 = \boldsymbol{r}_1 + \boldsymbol{s}$$

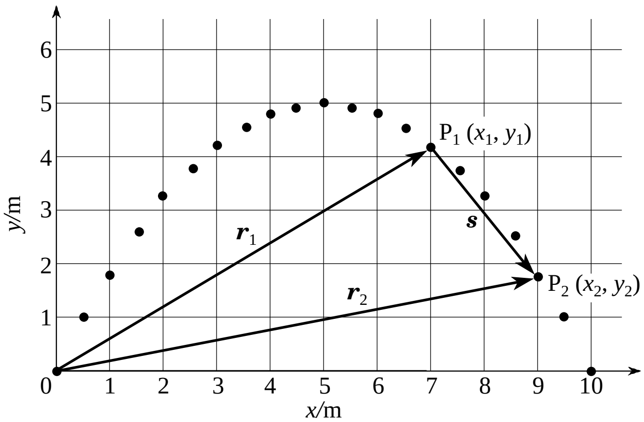

Figure 5 Displacement and position vectors.

The operation of adding vectors together is called **vector addition**, the process is not restricted to displacements but it can only be applied to vectors of 'similar type', e.g. we can add a velocity to a velocity, or an acceleration to an acceleration, but we cannot add a velocity to an acceleration.

A vector sum such as $\boldsymbol{r}_1 + \boldsymbol{s}$ looks a lot like an ordinary (scalar) sum, but it is really quite different because the *directions* of the vectors must be taken into account as well as their magnitudes. When thinking about vector addition you should have in mind a picture something like that shown in Figure 6, the essential features of which are described by the following rule:

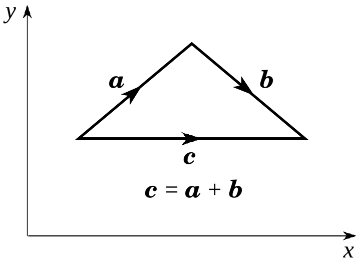

Figure 6 The triangle rule for adding vectors. Note the directions of the arrows: the diagram would be incorrect if any one, or any two of the arrows were reversed.

> The **triangle rule** for adding vectors:
>
> Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be vectors of similar type represented by appropriate arrows (or directed line segments). If the arrow representing $\boldsymbol{b}$ is drawn from the head of the arrow representing $\boldsymbol{a}$, then an arrow from the tail of $\boldsymbol{a}$ to the head of $\boldsymbol{b}$ represents the vector sum $\boldsymbol{c} = \boldsymbol{a} + \boldsymbol{b}$.

❖ If $\boldsymbol{c} = \boldsymbol{a} + \boldsymbol{b}$, will it necessarily be the case that $|\boldsymbol{c}| = |\boldsymbol{a}| + |\boldsymbol{b}|$?

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Although it is helpful to have a picture in mind when *thinking about* vector addition, it is usually easier to *perform* such additions with the aid of components. For instance, looking at Figure 6 you should be able to convince yourself that if $\boldsymbol{a} = (a_x, a_y)$, $\boldsymbol{b} = (b_x, b_y)$ and $\boldsymbol{c} = (c_x, c_y)$ then the vector equation $\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{c}$ may be written in the form

$$(a_x, a_y) + (b_x, b_y) = (c_x, c_y)$$

where $c_x = a_x + b_x$

and $c_y = a_y + b_y$

In other words,

> To add together vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, simply add the corresponding components so that:
>
> $$\boldsymbol{a} + \boldsymbol{b} = (a_x, a_y) + (b_x, b_y) = (a_x + b_x, a_y + b_y) \tag{7}$$

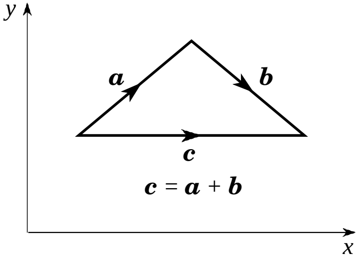

Figure 6 The triangle rule for adding vectors. Note the directions of the arrows: the diagram would be incorrect if any one, or any two of the arrows were reversed.

❖ If $\boldsymbol{c} = \boldsymbol{a} + \boldsymbol{b}$, where $\boldsymbol{a} = (1.6\,\text{m}, -5.5\,\text{m})$ and $\boldsymbol{b} = (-1.6\,\text{m}, 5.3\,\text{m})$, express $\boldsymbol{c}$ in terms of its components and describe its direction in words.

Apart from adding vectors another useful mathematical operation that can be carried out is that of multiplying a vector by a scalar. This is called **scaling** and may be defined in the following way

> To multiply a vector $\boldsymbol{a}$ by a scalar λ, simply multiply each component of $\boldsymbol{a}$ by λ so that:
>
> $$\lambda\boldsymbol{a} = \lambda(a_x, a_y) = (\lambda a_x, \lambda a_y) \tag{8}$$

The result of scaling $\boldsymbol{a}$ by λ is to produce a vector $\lambda\boldsymbol{a}$ of magnitude $|\lambda\boldsymbol{a}| = |\lambda||\boldsymbol{a}|$ that points in the same direction as $\boldsymbol{a}$ if λ is positive, and in the opposite direction to $\boldsymbol{a}$ if λ is negative. So, for example, given a displacement $\boldsymbol{a}$, the scaled vector $2\boldsymbol{a}$ would be twice as long and would point in the same direction, and the scaled vector $-\boldsymbol{a} = (-1)\boldsymbol{a}$ would have the same length as $\boldsymbol{a}$ but would point in the opposite direction (i.e. it would be **antiparallel**) to $\boldsymbol{a}$.

❖ If $\boldsymbol{a} = (-7\,\text{m}, 4\,\text{m})$ and $\boldsymbol{b} = (-1\,\text{m}, 2\,\text{m})$ what are the components of the vector $\boldsymbol{a} - 4\boldsymbol{b}$? What is the magnitude of $\boldsymbol{a} - 4\boldsymbol{b}$?

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Now that you know how to add and scale vectors you should be able to rearrange vector equations. For instance, referring back to Figure 5 and recalling that $\boldsymbol{r}_1 = (x_1, y_1)$, $\boldsymbol{r}_2 = (x_2, y_2)$ and $\boldsymbol{s} = (x_2 - x_1, y_2 - y_1)$, it is easy to see that the vector equation $\boldsymbol{r}_2 = \boldsymbol{r}_1 + \boldsymbol{s}$ can be rearranged to give a purely vectorial definition of displacement:

$$\boldsymbol{s} = \boldsymbol{r}_2 - \boldsymbol{r}_1$$

Given the rule for *adding* vectors, it should be pretty clear what we mean by a **vector difference** such as $\boldsymbol{r}_2 - \boldsymbol{r}_1$ but if you are in any doubt just regard the vector difference as the *sum* of $\boldsymbol{r}_2$ and the scaled vector $(-1)\boldsymbol{r}_1$. It then follows from the above definitions that

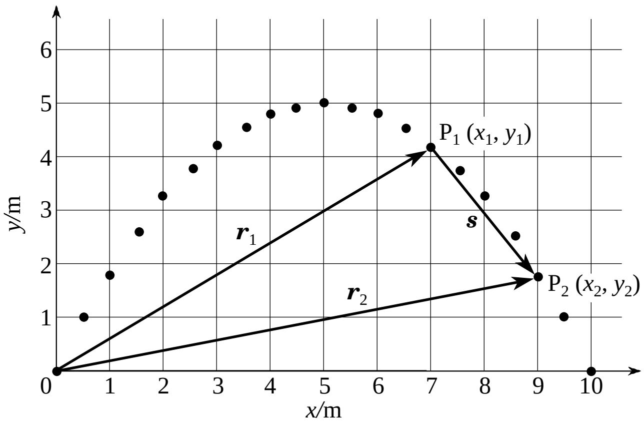

Figure 5 Displacement and position vectors.

$$\boldsymbol{s} = \boldsymbol{r}_2 - \boldsymbol{r}_1 = (x_2, y_2) - (x_1, y_1) = (x_2 - x_1, y_2 - y_1) \tag{9}$$

As you can see, vector equations can be treated very much like ordinary equations as far as rearrangements are concerned, though you should keep in mind the directional nature of vectors so that you avoid trying to do something silly like dividing by a vector.

Another possible rearrangement of $\boldsymbol{r}_2 = \boldsymbol{r}_1 + \boldsymbol{s}$ is

$$\boldsymbol{r}_2 - \boldsymbol{r}_1 - \boldsymbol{s} = \mathbf{0}$$

where $\mathbf{0} = (0, 0)$ represents the **zero vector** which has zero magnitude. (Note that a vector cannot be equal to a scalar, so you should always write $\boldsymbol{a} - \boldsymbol{a} = \mathbf{0}$ rather than $\boldsymbol{a} - \boldsymbol{a} = 0$.)

Question T2

A body moves from a point A, with position coordinates (4 m, 2 m), to a point B, with position coordinates (7 m, −3 m). What is the magnitude and direction of this displacement? Check your answers by drawing a scale diagram. ❏

2.3 Velocity in projectile motion

Let us return yet again to the kind of projectile motion shown in Figure 5, but let us now associate values of the time with various points in the motion. In particular suppose that at time t the projectile passes through a point P with position vector $\boldsymbol{r} = (x, y)$, and that a short time later, at $t + \Delta t$, the projectile arrives at a point with position coordinates $(x + \Delta x, y + \Delta y)$. ☞ We can then represent the change in position over the short time Δt by the displacement vector $\Delta\boldsymbol{r} = (\Delta x, \Delta y)$ and we can define the **average velocity** $\langle\boldsymbol{v}\rangle$ of the projectile as it moves between the two points by

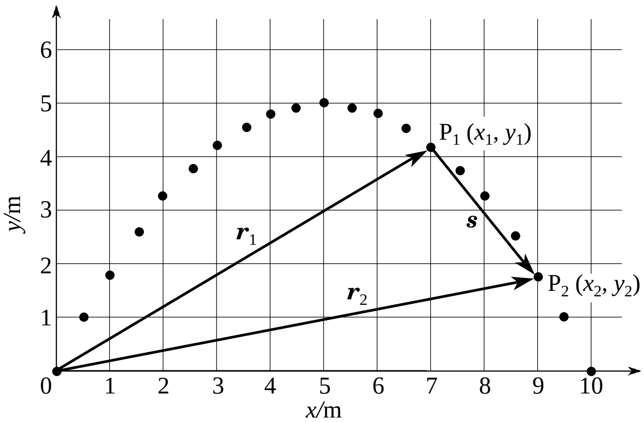

Figure 5 Displacement and position vectors.

$$\langle\boldsymbol{v}\rangle = \frac{\Delta\boldsymbol{r}}{\Delta t} = \frac{1}{\Delta t}(\Delta x, \Delta y) = \left(\frac{\Delta x}{\Delta t}, \frac{\Delta y}{\Delta t}\right) \qquad (10)$$ ☞

Note that the operation of dividing the two-dimensional displacement $\Delta\boldsymbol{r}$ by Δt really amounts to *scaling* $\Delta\boldsymbol{r}$ by $1/\Delta t$, so the resulting average velocity is another two-dimensional vector that points in the same direction as $\Delta\boldsymbol{r}$. The x-component of $\langle\,\boldsymbol{v}\,\rangle$ is given by $\Delta x/\Delta t$ and represents the average rate of change of the x-coordinate of the projectile, while the y-component, $\Delta y/\Delta t$, represents the average rate of change of the projectile's y-coordinate. This definition of the average velocity vector provides the natural two-dimensional generalization of the definition given elsewhere in *FLAP* for average velocity in one-dimensional (*linear*) motion. In effect the two-dimensional velocity can be regarded as the result of two independent linear velocities, one in the x-direction and the other in the y-direction.

In practice we often need to know the velocity of a projectile at a particular instant, rather than the average velocity over some specified interval. This **instantaneous velocity** $\boldsymbol{v}$ is also a vector quantity, the components of which, v_x and v_y, can be found by considering the corresponding components of average velocities taken over smaller and smaller intervals around the time in question. In mathematical terms we can indicate that the instantaneous velocity is a limiting case of the average velocity as the time interval Δt becomes vanishingly small by writing

$$\boldsymbol{v} = \lim_{\Delta t \to 0}\left(\frac{\Delta\boldsymbol{r}}{\Delta t}\right) = \lim_{\Delta t \to 0}\left(\frac{\Delta x}{\Delta t}, \frac{\Delta y}{\Delta t}\right) = \left[\lim_{\Delta t \to 0}\left(\frac{\Delta x}{\Delta t}\right), \lim_{\Delta t \to 0}\left(\frac{\Delta y}{\Delta t}\right)\right] \qquad (11)$$

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Now, if you are familiar with *calculus* you will know that *limits* of this kind are represented symbolically by *derivatives*, so we have

$$\boldsymbol{v} = \frac{d\boldsymbol{r}}{dt} = \left(\frac{dx}{dt}, \frac{dy}{dt}\right) \tag{12}$$

and since we can write $\boldsymbol{v} = (v_x, v_y)$ we can see that

$$v_x = \frac{dx}{dt} \quad \text{and} \quad v_y = \frac{dy}{dt}$$

Whether you are familiar with calculus or not ☞, you should be able to appreciate that these final equations have a simple graphical interpretation. If you were to draw a graph showing how the x-coordinate of the projectile changed with time (i.e. an x against t graph) then the *gradient* (i.e. slope) of that graph at any particular value of t would represent the instantaneous velocity component v_x at that time. Similarly, the gradient of a y against t graph at any particular value of t would represent the value of v_y at that time. Thus, each component of the projectile's instantaneous velocity represents the instantaneous rate of change of the corresponding component of the projectile's position vector.

It follows that the instantaneous velocity itself is the instantaneous rate of change of the position vector and at any particular time it is tangential to the trajectory (see Figure 7).

In discussing the velocity ☞ of a projectile we have, so far, emphasized the use of components but, as you know, a vector can also be characterized by its magnitude and direction. The magnitude of the (instantaneous) velocity $\boldsymbol{v}$ is called the (instantaneous) **speed** and may be represented by v or $|\boldsymbol{v}|$: it is of course a *scalar* quantity and it can never be negative. Thus, it makes perfectly good sense to say that the velocity component $v_y = -2.4\,\mathrm{m\,s^{-1}}$, but it would be utter nonsense to say that the speed v had the same value. For the two-dimensional motion of a projectile, the direction of the velocity $\boldsymbol{v}$ can be described by the angle ϕ (measured in the anticlockwise direction) between the positive x-axis and the velocity vector (see Figure 7).

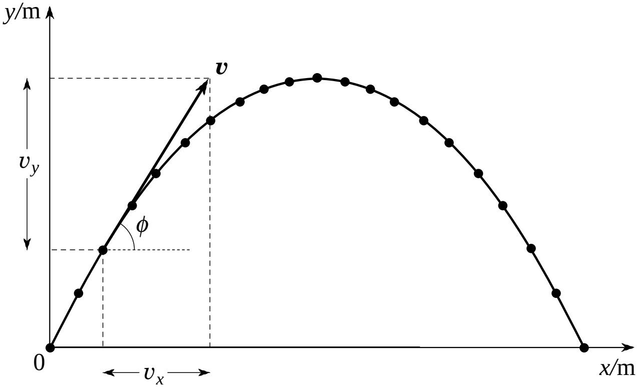

Figure 7 The instantaneous velocity $v = (v_x, v_y)$ of a projectile is tangential to the trajectory at any point.

So, given a velocity $\boldsymbol{v}$ we can write down the following relationships

$$\boldsymbol{v} = (v_x, v_y)$$

with $$v = |\boldsymbol{v}| = \sqrt{v_x^2 + v_y^2} \quad \text{and} \quad \tan\phi = \frac{v_y}{v_x}$$

or $$v_x = v\cos\phi \quad \text{and} \quad v_y = v\sin\phi$$

Study comment Notice that in Figure 7 we have used ϕ to represent the angle between $\boldsymbol{v}$ and the x-axis, whereas in Figure 3 we used θ to represent the angle between $\boldsymbol{r}$ and the x-axis. In problems involving projectile motion we must be careful *never to confuse these two angles*, which are usually quite different. In this module we will reinforce the distinction by always using the appropriate symbol for each angle.

Question T3

Find the magnitude and direction of a velocity which has an x-component, v_x, of $8\,\mathrm{m\,s^{-1}}$ and which has a y-component, v_y, of $10\,\mathrm{m\,s^{-1}}$. ❏

Figure 8 is an enlarged version of Figure 3, without the position vector. We can use this diagram to investigate the motion of the projectile in the x- and y-directions. The dots represent successive positions of the projectile at intervals of 0.1 s.

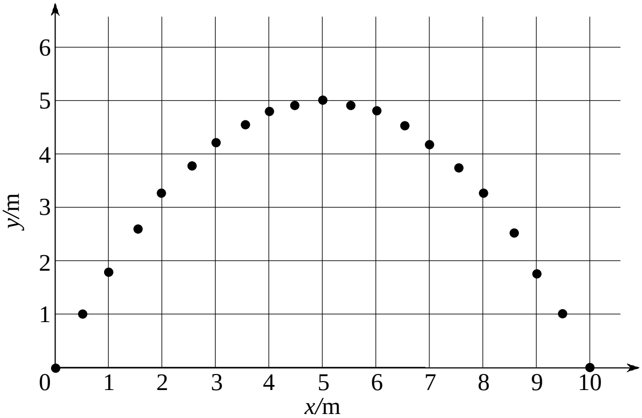

Figure 8 Successive positions of a projectile at intervals of 0.1 s.

Question T4

Using Figure 8, measure the x-component of the displacement between the first and second, second and third, eighth and ninth, fourteenth and fifteenth, and sixteenth and seventeenth dots. What do your answers suggest about the x-component of the projectile's velocity? ❏

It seems from the solution to Question T4 that the horizontal component of the velocity of a projectile is constant.

We can also use Figure 8 to try to acquire information about the vertical component of the velocity of a projectile.

Question T5

Using Figure 8, measure the y-component of displacement between the same points as in Question T4. What do these values suggest about the y-component of the projectile's velocity? ❑

From the solution to Question T5, the vertical component of the velocity, v_y, is *not* constant. Its value is positive at the launch point but decreases as the projectile ascends, $v_y = 0$ momentarily when the projectile reaches its maximum altitude and thereafter v_y is negative and decreasing until the projectile hits the ground. Clearly, if we want to describe the velocity of the projectile more precisely we need some way of describing the rate of change of this vertical component of velocity; that is the function of the next subsection.

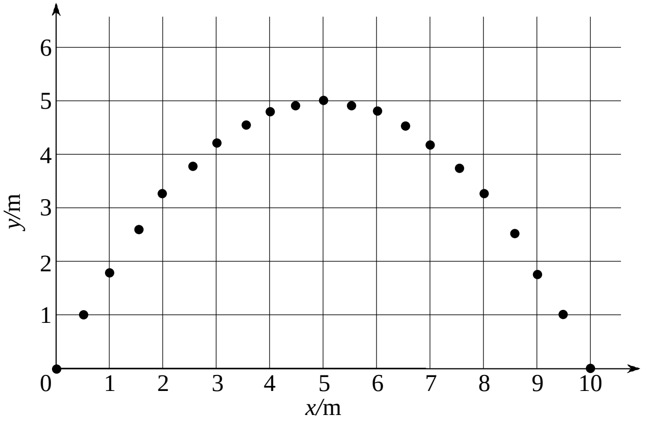

Figure 8 Successive positions of a projectile at intervals of 0.1 s.

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2.4 Acceleration in projectile motion

Just as we can use a two-dimensional velocity to describe the rate of change of position of the projectile so we can use a two-dimensional **instantaneous acceleration** to describe the rate of change of the projectile's velocity. So, if the velocity of the projectile changes from $\boldsymbol{v}$ at time t, to $\boldsymbol{v} + \Delta\boldsymbol{v}$ at time $t + \Delta t$, we can say that the instantaneous acceleration at time t is

$$\boldsymbol{a} = \lim_{\Delta t \to 0}\left(\frac{\Delta\boldsymbol{v}}{\Delta t}\right) = \lim_{\Delta t \to 0}\left(\frac{\Delta v_x}{\Delta t}, \frac{\Delta v_y}{\Delta t}\right) = \left[\lim_{\Delta t \to 0}\left(\frac{\Delta v_x}{\Delta t}\right), \lim_{\Delta t \to 0}\left(\frac{\Delta v_y}{\Delta t}\right)\right]$$

or, in terms of derivatives

$$\boldsymbol{a} = \frac{d\boldsymbol{v}}{dt} = \left(\frac{dv_x}{dt}, \frac{dv_y}{dt}\right) \tag{13}$$

In terms of components, $\boldsymbol{a} = (a_x, a_y)$

where $\quad a_x = \dfrac{dv_x}{dt} \quad$ and $\quad a_y = \dfrac{dv_y}{dt}$

As usual we can interpret these equations graphically: the value of a_x at any particular time is given by the gradient of the υ_x against t graph at that particular time, and similarly for υ_y.

❖ Express the magnitude of the instantaneous acceleration $\boldsymbol{a}$ in terms of its components a_x and a_y, and then express the components in terms of the magnitude and the angle ψ ☞ from the positive x-axis to $\boldsymbol{a}$.

In what follows we will generally refer to the *instantaneous acceleration* simply as the *acceleration*.

You learned in the last subsection that one characteristic of projectile motion is that the horizontal component of velocity, υ_x, is constant. This means that the rate of change of υ_x must be zero and consequently $a_x = 0$.

You also learned that the vertical component of the velocity changes continuously throughout the motion. Careful measurements would show that in the absence of air resistance υ_y *decreases at a constant rate*. In other words, throughout the motion the vertical component of velocity is reduced by equal amounts $\Delta\upsilon_x$ in equal intervals of time Δt, irrespective of when those intervals begin and end. It follows that the vertical component of acceleration is a negative constant which we can write as $a_y = -g$. Since this vertical component of acceleration is caused by the action of gravity on the projectile we say that the constant g is the **magnitude of the acceleration due to gravity**.

The value of g varies from place to place over the Earth's surface, but it is generally about 9.8 m s^{-2} and is usually taken to be 9.81 m s^{-2} throughout the UK. It describes the acceleration with which all objects near the surface of the Earth fall, so long as they are not impeded appreciably by air resistance.

Combining the observations that $a_x = 0$ and $a_y = -g$ we have:

$$\boldsymbol{a} = (a_x, a_y) = (0, -g) \tag{14}$$

and $$a = |\boldsymbol{a}| = g$$

It is this particular acceleration that characterizes a projectile near the Earth's surface (in the absence of air resistance).

✥ If you were to measure the speed of a projectile at two different times separated by an interval of one second would you always expect them to differ by about 9.8 m s^{-1}? If not, under what conditions would you expect the speeds to differ by about 9.8 m s^{-1}. (As usual, ignore air resistance, but think carefully!)

2.5 The independence of *x*- and *y*-motions for projectiles

As we have seen in Subsections 2.2 and 2.3:

> The two key features of projectile motion are:
>
> 1 A constant horizontal velocity component, v_x (since $a_x = 0$).
>
> 2 A constant acceleration $\boldsymbol{a}$ of magnitude $g = 9.81\,\mathrm{m\,s^{-2}}$ directed vertically downwards, i.e. $a_y = -9.81\,\mathrm{m\,s^{-2}}$, if upwards is taken as the direction of increasing y.

These two features will help you to solve a vast range of projectile problems provided you keep one other principle in mind:

> The horizontal and vertical motions of a projectile must have the same duration, but are otherwise independent of one another and can be treated as two separate one-dimensional (linear) motions.

The idea that movement in the x-direction is independent of movement in the y-direction may sound simple and obvious, but many students find it somewhat counter-intuitive and are led into making simple mistakes by forgetting it. The following question is one that many who have not been forewarned might get wrong.

❖ Suppose you have two identical bullets and you drop one while firing the other horizontally from a high velocity rifle. Which bullet will hit the ground first? (Ignore air resistance, as usual.)

3 Applying the equations of motion

3.1 Horizontal motion

In Subsection 2.4 we saw that a projectile does not experience any acceleration in the horizontal direction. So, in terms of our usual Cartesian coordinate system

$$a_x = 0 \qquad (15)$$

It follows that the *x*-component of the projectile's velocity, v_x, is constant and will therefore be equal to its initial value at the launch point. So, if this initial value of the horizontal velocity component is denoted by u_x, we can write

$$v_x = u_x \qquad (16)$$

It follows from this that if the projectile is launched at time $t = 0$, then at time t the x-component of its displacement from the launch point will be

$$s_x = u_x t \tag{17}$$

$$a_x = 0 \quad \text{(Eqn 15)}$$

$$v_x = u_x \quad \text{(Eqn 16)}$$

Equations 15, 16 and 17 are called the **uniform motion equations**. They are introduced elsewhere in *FLAP* in the context of one-dimensional (linear) motion, but they are also relevant here because of the independence of the horizontal and vertical motions, and the lack of horizontal acceleration.

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3.2 Vertical motion

In Subsection 2.4 we also stressed that the vertical motion of a projectile is determined by the constant vertical acceleration due to gravity. So, in terms of our usual Cartesian coordinate system

$$a_y = -g \tag{18}$$

Since this implies that the vertical component of velocity v_y decreases at a constant rate from its initial value u_y we can say that at time t after launch

$$\boxed{v_y = u_y - gt} \tag{19}$$

Now, since v_y is decreasing at a *constant* rate, its average value between the moment of launch ($t = 0$) when it has the initial value u_y and any later time t when it has the final value v_y will be $\langle v_y \rangle = (u_y + v_y)/2$. It follows that at time t the vertical component of the projectile's displacement from its launch site will be

$$s_y = \langle v_y \rangle t = \left(\frac{u_y + v_y}{2}\right)t = \left(\frac{u_y + u_y - gt}{2}\right)t$$

where Equation 19 has been used to eliminate v_y in the last step. Thus

$$\boxed{s_y = u_y t - \tfrac{1}{2}gt^2} \tag{20}$$

We can relate v_y to s_y by first squaring both sides of Equation 19

$$v_y = u_y - gt \qquad \text{(Eqn 19)}$$

to obtain

$$v_y^2 = u_y^2 - 2u_y gt + g^2t^2 = u_y^2 - 2g\left(u_y t - \frac{1}{2}gt^2\right)$$

and then using Equation 20

$$s_y = u_y t - \tfrac{1}{2}gt^2 \qquad \text{(Eqn 20)}$$

to replace the expression in brackets by s_y. Thus

$$v_y^2 = u_y^2 - 2gs_y \qquad (21)$$

Equations 19, 20 and 21 (together with Equations 15, 16 and 17) are the main equations used to solve projectile problems.

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Equations 19, 20 and 21 are in fact a special case (corresponding to $a_y = -g$) of the **uniform acceleration equations** introduced elsewhere in *FLAP*

$$v_y = u_y + a_y t \tag{22}$$

$$s_y = u_y t + \tfrac{1}{2} a_y t^2 \tag{23}$$

$$v_y^2 = u_y^2 + 2a_y s_y \tag{24}$$

These are slightly more general than Equations 19, 20 and 21 since they apply for *any* constant value of a_y, though it should be noted that they do *not* describe more general situations in which a_y varies with time or position. We will return to Equations 22, 23 and 24 later since, together with Equations 15 to 17, they will allow us to extend our treatment of projectile problems to cover any form of two-dimensional motion where there is uniform acceleration along the y-axis and no acceleration along the x-axis.

When using any of Equations 15 to 24 it is important to remember that the displacements s_x and s_y are measured from the position of the body at time $t = 0$.

Question T6

A ball is thrown vertically upwards with a velocity of $20\,\mathrm{m\,s^{-1}}$. How long will the ball take to reach the highest point before it begins to fall and what is this maximum height above the point of launch ?
(Assume $g = 9.81\,\mathrm{m\,s^{-2}}$.) ❏

3.3 The trajectory of a projectile

The shape of a projectile's trajectory may be specified in a number of ways, for example Equations 17 and 20 express the displacement components s_x and s_y in terms of the time t since launch — thus providing a *parametric* description of the trajectory. However, a more informative specification may be obtained by expressing s_y directly in terms of s_x. We can derive this relationship by eliminating t from Equations 17 and 20 as follows.

From Equation 17 $\quad t = \dfrac{s_x}{u_x}$

Substituting this into Equation 20 gives us

$$s_y = u_y\left(\frac{s_x}{u_x}\right) - \frac{1}{2}g\left(\frac{s_x}{u_x}\right)^2$$

This equation can be rearranged to give

$$s_y = \frac{u_y}{u_x}s_x - \frac{g}{2u_x^2}s_x^2 \qquad (25)$$

Any projectile, launched with speed u at an angle ϕ to the x-axis, will initially have velocity components $u_x = u\cos\phi$ and $u_y = u\sin\phi$, so Equation 25

$$s_y = \frac{u_y}{u_x}s_x - \frac{g}{2u_x^2}s_x^2 \qquad \text{(Eqn 25)}$$

becomes

$$s_y = \frac{u\sin\phi}{u\cos\phi}s_x - \frac{g}{2u^2\cos^2\phi}s_x^2 = s_x\tan\phi - \frac{g}{2u^2\cos^2\phi}s_x^2 \qquad (26)$$

Since u and ϕ are constants we can write Equation 26 in the form

$$s_y = As_x - Bs_x^2 \qquad (27)$$

where A and B are constants. Equations 25 and 27 show that s_y is a *quadratic function* of s_x, and imply that the graph of s_y against s_x will have the shape known as a **parabola**. The projectile trajectories shown in Figures 3, 5 and 8 were all parabolas.

A projectile moving near the Earth, under the influence of gravity alone, follows a parabolic trajectory.☞

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3.4 The range of a projectile

Archers, gunners and cricketers often want to know how to maximize the **range** of a projectile, i.e. the horizontal distance between the launching and landing points. This can be easily deduced from Equation 25 (or 26) if the launching and landing points are at the same vertical height because under these conditions the final vertical displacement will be $s_y = 0$ and Equation 25 will give

$$0 = \frac{u_y}{u_x}s_x - \frac{g}{2u_x^2}s_x^2 \qquad \text{(Eqn 25)}$$

i.e. $$\frac{u_y}{u_x}s_x = \frac{g}{2u_x^2}s_x^2$$

One solution to this equation is $s_x = 0$, which corresponds to the launching of the projectile. However, this mathematical solution is not of much physical interest. The solution we want (corresponding to the landing of the projectile) has $s_x \neq 0$. We can therefore divide both sides of the equation by s_x and rearrange to obtain

$$s_x = \frac{2u_y u_x}{g}$$

This value of s_x is the horizontal displacement from the point of projection (the launch) to the landing point.

The magnitude of this displacement is the range, R. Therefore

$$R = \left| \frac{2u_y u_x}{g} \right| \qquad (28)$$

☞

We are now in a position to consider a problem which is of interest to cricketers or rounders players. At what angle to the horizontal, ϕ, should a fielder throw a ball to achieve maximum range?

Figure 9 defines the range

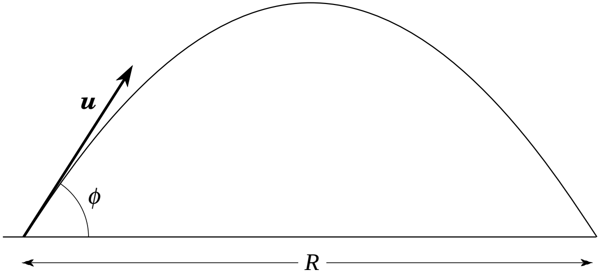

Figure 9 The range of a thrown ball.

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while Figure 10 shows the effect on the range of varying the angle of projection. If the ball is thrown too steeply it achieves plenty of height but very little range. On the other hand, if it thrown at too shallow an angle, gravity will pull the ball down before it has chance to travel far. There must be some intermediate value of ϕ at which the maximum range is achieved.

To simplify the problem, we will assume that the ball is thrown from ground level at the same speed whatever the angle and that air resistance is negligible.

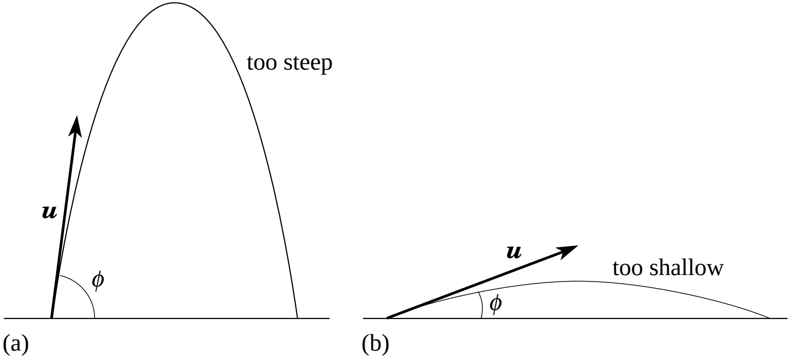

Figure 10 Maximizing the range of a projectile.

Suppose the ball is thrown with an initial velocity $\boldsymbol{u}$ at an angle ϕ to the horizontal, achieving a range R. The initial (and subsequent) horizontal component of velocity is $u_x = u\cos\phi$ and the initial vertical component of velocity is $u_y = u\sin\phi$. Substituting these values into Equation 28

$$R = \left|\frac{2u_y u_x}{g}\right| \qquad \text{(Eqn 28)}$$

gives us

$$R = \left|\frac{2u^2\sin\phi\cos\phi}{g}\right| \qquad (29)$$

Using the trigonometric identity

$$2\sin\phi\cos\phi = \sin(2\phi)$$

☞

Since u^2 and g are positive quantities we can write Equation 29 as

$$R = \frac{u^2}{g}\left|\sin(2\phi)\right| \qquad (30)$$

☞

❖ Assuming ϕ is between 0° and 180°, what are the greatest and least values that $\sin(2\phi)$ can have, and hence what is the maximum range?

We now need to find the angle of projection, ϕ, which produces the maximum range.

❖

What is the value of ϕ that gives the maximum range? ☞

A projectile moving near the Earth, under the influence of gravity alone, achieves maximum horizontal range when launched at 45° to the horizontal. ☞

For the flight of many objects, such as would be used in cricket or in putting the shot, these assumptions are fairly good.

Question T7

In the 1976 Olympic Games the shot-putting event was won with a throw of 21.32 m. Assuming that the shot was thrown from ground level at the optimum angle (45°), calculate the speed at which it was projected, making use of the formula for range. (Assume $g = 9.81\,\mathrm{m\,s^{-2}}$.) ❏

In the next section you will learn how to solve similar problems in a more fundamental way, without resorting to the range formula.

4 Solving projectile problems

Using the ideas developed in the previous two sections you should be able to solve almost any projectile problem. It is particularly important to remember that:

> In projectile motion, the horizontal component of velocity is constant and the vertical component of acceleration is constant. ☞

The usual strategy for dealing with projectile problems is as follows:

1. Express the initial velocity $\boldsymbol{u}$ in terms of its horizontal and vertical components $u_x = u\cos\phi$ and $u_y = u\sin\phi$.
2. Treat each component of the motion as an example of one-dimensional (linear) motion: one along a horizontal line at a constant velocity, v_x, and the other along a vertical line at a constant acceleration $-g$, (assuming vertically upwards to be the direction in which y increases).
3. Link the two component motions together by means of the total time of flight, T, which must be the same for each.

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4.1 Some examples of projectile motion

Before you tackle a projectile problem on your own, here is a worked example.

Example 1 A cannon points at 60° above the horizontal and fires a ball from ground level with a muzzle speed of $20.0\,\mathrm{m\,s^{-1}}$. Find the horizontal range R, the maximum height h and the time of flight T. Neglect air resistance.

Solution Using the strategy described above to solve this problem, we will first find the horizontal and vertical components of the initial velocity, u_x and u_y, respectively.

$$u_x = (20.0 \times \cos 60°)\,\mathrm{m\,s^{-1}} = 10.0\,\mathrm{m\,s^{-1}}$$

and $$u_y = (20.0 \times \sin 60°)\,\mathrm{m\,s^{-1}} = 17.3\,\mathrm{m\,s^{-1}}$$

We can now make use of the two properties of projectile motion.

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Horizontal motion The displacement s_x is given by Equation 17

$$s_x = u_x t \qquad \text{(Eqn 17)}$$

The horizontal range R can be found from this equation by putting $t = T$, (where T denotes the time of flight) and taking the modulus.

Therefore $$R = |u_x T| \qquad (31)$$

We can find T by considering the vertical motion.

Vertical motion When the ball lands on the ground at the end of the flight $t = T$ and the final vertical displacement is $s_y = 0$. So, upon substituting $s_y = 0$ and $t = T$ into

$$s_y = u_y t - \tfrac{1}{2} g t^2 \qquad \text{(Eqn 20)}$$

we have

$$0 = u_y T - \tfrac{1}{2} g T^2 = T\left(u_y - \tfrac{1}{2} g T\right)$$

This is a *quadratic equation* and therefore has two solutions ☞. These are

$$T = 0 \text{ and } T = \frac{2u_y}{g} = \frac{2 \times 17.3\,\mathrm{m\,s^{-1}}}{9.81\,\mathrm{m\,s^{-2}}} = 3.53\,\mathrm{s} \qquad (32)$$

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The first solution, $T = 0$, is the time the ball leaves the cannon; the second solution corresponds to the ball hitting the ground again and this is the time we need to find the range R.

Now that we have found the time of flight we can substitute this value for T into Equation 31,

$$R = |u_xT| \qquad \text{(Eqn 31)}$$

giving $R = |u_xT| = 10\,\mathrm{m\,s^{-1}} \times 3.53\,\mathrm{s} = 35.3\,\mathrm{m}$.

When the ball is at its maximum height, $s_y = h$, and the vertical component of velocity is momentarily zero, so $\upsilon_y = 0$. We can substitute these values into Equation 21

$$\upsilon_y^2 = u_y^2 - 2gs_y \qquad \text{(Eqn 21)}$$

to give the maximum height

$$0 = u_y^2 - 2gh$$

i.e. $h = \dfrac{u_y^2}{2g} = \dfrac{(17.3\,\mathrm{m\,s^{-1}})^2}{2 \times 9.81\,\mathrm{m\,s^{-2}}} = 15.3\,\mathrm{m}$ ❏

Now apply this strategy yourself to the next two questions.

Question T8

A cannon standing on the top of a cliff, 40 m high, fires a cannonball horizontally out to sea with a muzzle speed of $140\,\mathrm{m\,s^{-1}}$. How far out to sea does the ball go? (Assume $g = 9.81\,\mathrm{m\,s^{-2}}$.) ❏

Question T9

A golf ball is hit on level ground so that it leaves the ground with an initial velocity of $40.0\,\mathrm{m\,s^{-1}}$ at 30° above the horizontal. Find the greatest height reached by the ball, the total time of flight of the ball and the range of the shot. Neglect air resistance and assume $g = 9.81\,\mathrm{m\,s^{-2}}$. ❏

4.2 The vector equations for motion with uniform acceleration

Study comment This subsection develops vector expressions for the equations of motion of a projectile. Familiarity with these expressions is not necessary in order to meet the achievements of this module, but they may help you to become more familiar with the use of vectors.

As indicated earlier, the methods developed in this module cannot only solve projectile problems, but any problem involving motion in two dimensions in which there is constant acceleration (including zero acceleration). So far, we have always chosen our axes so that there is no acceleration in the x-direction, and consequently $a_x = 0$. When dealing with the general vector equations of uniformly accelerated two-dimensional motion, it does no harm to consider a more general case in which the x- and y-axes point in arbitrary directions in the plane of motion. In this general, case there will be uniform acceleration along *both* axes, so applying Equations 22, 23 and 24 in each case

$$v_x = u_x + a_x t \quad (33)$$

and $$v_y = u_y + a_y t \quad \text{(Eqn 22)}$$ ☞

We can write these as a single *vector* equation by using the rules for adding and scaling vectors that were introduced in Subsection 2.1

$$\boldsymbol{v} = (v_x, v_y) = (u_x + a_xt, u_y + a_yt) = (u_x, u_y) + (a_x, a_y)t$$

i.e. $\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{a}t$ (34)

and in the same way we can write

$$s_x = u_xt + \tfrac{1}{2}a_xt^2 \qquad (35)$$

and $$s_y = u_yt + \tfrac{1}{2}a_yt^2 \qquad \text{(Eqn 23)}$$

as the single *vector* equation $\boldsymbol{s} = \boldsymbol{u}t + \tfrac{1}{2}\boldsymbol{a}t^2$ (36)

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The two-component equations

$$v_x^2 = u_x^2 + 2a_x s_x \tag{37}$$

and $$v_y^2 = u_y^2 + 2a_y s_y \qquad \text{(Eqn 24)}$$

can be summed to give

$$v^2 = v_x^2 + v_y^2 = u_x^2 + u_y^2 + 2(a_x s_x + a_y s_y)$$

which can be combined into a single equation

$$v^2 = u^2 + 2\boldsymbol{a}\cdot\boldsymbol{s} \tag{38}$$

where the symbol $\boldsymbol{a}\cdot\boldsymbol{s}$ represents the **scalar product** of the vectors $\boldsymbol{a}$ and $\boldsymbol{s}$ which (in two dimensions) is defined by

$$\boldsymbol{a}\cdot\boldsymbol{s} = a_x s_x + a_y s_y$$

☞

Note that it follows from this definition that the scalar product of two vectors is a scalar, so that Equation 38

$$v^2 = u^2 + 2\boldsymbol{a} \cdot \boldsymbol{s} \tag{38}$$

is actually a scalar equation even though it is expressed in terms of vectors. It is also worth noting that the squared magnitudes v^2 and u^2 that appear in Equation 38 could both be expressed as scalar products since

$$v^2 = v_x^2 + v_y^2 = (\boldsymbol{v} \cdot \boldsymbol{v}) \quad \text{and} \quad u^2 = u_x^2 + u_y^2 = (\boldsymbol{u} \cdot \boldsymbol{u})$$

$$\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{a}t \qquad \text{(Eqn 34)}$$

$$\boldsymbol{s} = \boldsymbol{u}t + \tfrac{1}{2}\boldsymbol{a}t^2 \qquad \text{(Eqn 36)}$$

Now, Equations 34, 36 and 38 are not frequently used, indeed Equation 38 actually scrambles together some of the information that is contained in Equations 24 and 37.

and

$$v_y^2 = u_y^2 + 2a_y s_y \qquad \text{(Eqn 24)}$$

$$v_x^2 = u_x^2 + 2a_x s_x \qquad \text{(Eqn 37)}$$

Nonetheless, the vector equations do emphasize the remarkable power of vectors. In particular, if we were asked to solve a **three-dimensional** problem in which a particle was free to move with components in three independent directions simultaneously (call them x, y and z), subject to constant acceleration in any direction, we could immediately say that Equations 34, 36 and 38

$$\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{a}t \qquad \text{(Eqn 34)}$$

$$\boldsymbol{s} = \boldsymbol{u}t + \tfrac{1}{2}\boldsymbol{a}t^2 \qquad \text{(Eqn 36)}$$

$$v^2 = u^2 + 2\boldsymbol{a}\cdot\boldsymbol{s} \qquad \text{(Eqn 38)}$$

will describe the motion provided we interpret $\boldsymbol{s}$, $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{a}$ as *three-dimensional* vectors which can be represented by **ordered triples** such as $\boldsymbol{s} = (s_x, s_y, s_z)$, $\boldsymbol{u} = (u_x, u_y, u_z)$ and so on, and $\boldsymbol{a}\cdot\boldsymbol{s}$ is the *three-dimensional* scalar product defined by

$$\boldsymbol{a}\cdot\boldsymbol{s} = a_x s_x + a_y s_y + a_z s_z$$

So, vectors make it easy to write results in a form that is readily generalized to the three dimensions of the real world and are therefore of great value in physics.

Question T10

A puck is moving along the *y*-axis on an ice-rink (so that friction is negligible) at a speed of $10\,\mathrm{m\,s^{-1}}$. We define the *y*-axis along the surface of the rink, in the direction of the original motion of the puck. The *x*-axis is also on the surface of the rink and perpendicular to the *y*-axis. If the puck is subjected to an acceleration $\boldsymbol{a} = (2\,\mathrm{m\,s^{-2}}, 0, 0)$ for 5 s, in what direction will it finally be moving? At the end of the acceleration, what is the displacement of the puck from its position at the start of the acceleration? ❑

4.3 Solution to the introductory problem

Study comment This subsection contains the solution to the problem that was posed in Subsection 1.1. You should reread the problem and attempt to answer it before you work through the solution below.

We require the *minimum* muzzle speed. This corresponds to the launch velocity that will just enable the shell to reach the height of the aircraft. This means that the shell's maximum height must equal the height at which the aircraft is flying. Since the shell is fired when the aircraft is directly overhead, the horizontal component of the muzzle velocity must equal the horizontal velocity of the aircraft, so that it arrives at the aircraft's height with the correct horizontal displacement.

First consider the vertical motion. The vertical component of the initial velocity, u_y is $u \sin \phi$ and the vertical acceleration, a_y is $-9.81\,\mathrm{m\,s^{-2}}$. At the top point of the motion the displacement, s_y is $2000\,\mathrm{m}$ and the vertical component of velocity, υ_y is $0\,\mathrm{m\,s^{-1}}$.

Using $$\upsilon_y^2 = u_y^2 - 2gs_y \qquad \text{(Eqn 21)}$$

we have $$0 = u^2 \sin^2 \phi - 2gs_y$$

so $$u^2 \sin^2 \phi = 2gs_y = 2 \times 9.81\,\mathrm{m\,s^{-2}} \times 2000\,\mathrm{m} = 3.92 \times 10^4\,(\mathrm{m\,s^{-1}})^2 \qquad (39)$$

Now consider the horizontal motion. The horizontal constant velocity, v_x is equal to the horizontal component of the launch velocity, that is $u\cos\phi$. To keep pace with the aircraft this must equal $150\,\mathrm{m\,s^{-1}}$.

Therefore $\quad u\cos\phi = 150\,\mathrm{m\,s^{-1}}$

and $\quad u^2\cos^2\phi = 2.25\times10^4\,(\mathrm{m\,s^{-1}})^2 \qquad (40)$

If we add Equations 39 and 40 we obtain

$$u^2\sin^2\phi = 2gs_y = 2\times9.81\,\mathrm{m\,s^{-2}}\times2000\,\mathrm{m} = 3.92\times10^4\,(\mathrm{m\,s^{-1}})^2 \qquad \text{(Eqn 39)}$$

$$u^2(\sin^2\phi + \cos^2\phi) = (3.92 + 2.25)\times10^4\,(\mathrm{m\,s^{-1}})^2$$

Using the trigonometric identity

$$\sin^2\phi + \cos^2\phi = 1$$

we find $\quad u^2 = 6.17\times10^4\,(\mathrm{m\,s^{-1}})^2$

so $\quad u = 248\,\mathrm{m\,s^{-1}}$

Now we need to find the launch angle ϕ. If we divide each side of Equation 39 by the corresponding side of Equation 40

$$u^2 \sin^2 \phi = 2gs_y = 2 \times 9.81\,\mathrm{m\,s^{-2}} \times 2000\,\mathrm{m} = 3.92 \times 10^4\,(\mathrm{m\,s^{-1}})^2 \qquad \text{(Eqn 39)}$$

$$u^2 \cos^2 \phi = 2.25 \times 10^4\,(\mathrm{m\,s^{-1}})^2 \qquad \text{(Eqn 40)}$$

we find

$$\frac{\sin^2 \phi}{\cos^2 \phi} = \tan^2 \phi = \frac{3.92 \times 10^4\,(\mathrm{m\,s^{-1}})^2}{2.25 \times 10^4\,(\mathrm{m\,s^{-1}})^2} = 1.74$$

which gives $\phi = \arctan(1.32) = 52.9°$

For the shell to hit the aircraft, it should be fired with a minimum speed of $248\,\mathrm{m\,s^{-1}}$ (at an angle of 52.9° to the horizontal). It is worth noting that since v_x is equal to the speed of the aircraft, the shell will be travelling vertically beneath the plane until it hits! On impact, the velocities of aircraft and shell will be identical ($v_x = 150\,\mathrm{m\,s^{-1}}$, $v_y = 0$) and so the relative velocity between the two will be zero. Any damage to the aircraft will be due to the explosive charge, not the impact itself.

5 Closing items

5.1 Module summary

1 A *projectile* is an object that is launched into unpowered flight near the Earth's surface.

2 Some projectiles can be adequately represented by point *particles* that move under the influence of gravity alone. In this approximation the flight of a projectile is an example of *two-dimensional* motion under constant acceleration.

3 The location of a point in a plane can be specified by its *position coordinates* (x, y) relative to a two-dimensional Cartesian coordinate system, or by its two-dimensional *position vector*, $\boldsymbol{r}$. The position vector of a point can be specified in terms of its *magnitude* $r = |\boldsymbol{r}|$ (which represents the *distance* from the origin to the point) and its *direction* as given by the angle θ measured anticlockwise from the positive x-axis to $\boldsymbol{r}$. The position vector of a point can also be specified in terms of its *components* which are equal to the position coordinates of the point, thus we can write $\boldsymbol{r} = (x, y)$ with $r = |\boldsymbol{r}| = \sqrt{x^2 + y^2}$ and $\tan\theta = y/x$.

4 The *displacement* from a point with position vector $\boldsymbol{r}_1 = (x_1, y_1)$ to a point with position vector $\boldsymbol{r}_2 = (x_2, y_2)$ is defined by the vector quantity $\boldsymbol{s} = \boldsymbol{r}_2 - \boldsymbol{r}_1 = (x_2 - x_1, y_2 - y_1)$ and represents a change or difference in position. Unlike position vectors, displacements in a given system of coordinates are independent of any particular origin and may be measured from any selected reference point. In projectile motion the displacement of the projectile is usually measured from its launch point.

5 In general, *vector quantities* require both a magnitude and a direction for their complete specification and may be contrasted with *scalar quantities* which can be specified by a magnitude alone. In two-dimensions any vector $\boldsymbol{v}$ may be represented by an *ordered pair* of (scalar) components (v_x, v_y); the magnitude of $\boldsymbol{v}$ may then be written $v = |\boldsymbol{v}| = \sqrt{v_x^2 + v_y^2}$, and its direction may be specified by the angle ϕ given by $\tan\phi = v_y/v_x$. Under these circumstances $v_x = v\cos\phi$ and $v_y = v\sin\phi$.

6 Given two vectors of similar type the operation of *vector addition* allows us to add them graphically using the *triangle rule* or algebraically using

$$\boldsymbol{a} + \boldsymbol{b} = (a_x, a_y) + (b_x, b_y) = (a_x + b_x, a_y + b_y) \qquad \text{(Eqn 7)}$$

The operation of *scaling* allows us to multiply any vector by a scalar to produce another vector according to

$$\lambda\boldsymbol{a} = \lambda(a_x, a_y) = (\lambda a_x, \lambda a_y) \qquad \text{(Eqn 8)}$$

This has magnitude $|\lambda||\boldsymbol{a}|$ and points in the same direction as $\boldsymbol{a}$ if λ is positive, and in the opposite direction if λ is negative.

7 The velocity, $\boldsymbol{v}$, and acceleration, $\boldsymbol{a}$, of a particle moving in two dimensions are vector quantities that are defined as follows:

$$\boldsymbol{v} = (v_x, v_y) = \frac{d\boldsymbol{r}}{dt} = \left(\frac{dx}{dt}, \frac{dy}{dt}\right) \qquad \text{(Eqn 12)}$$

$$\boldsymbol{a} = (a_x, a_y) = \frac{d\boldsymbol{v}}{dt} = \left(\frac{dv_x}{dt}, \frac{dv_y}{dt}\right) \qquad \text{(Eqn 13)}$$

The magnitude of the velocity is called the *speed* and cannot be negative.

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8 Projectile motion is characterized by a constant horizontal velocity component, $v_x = u_x$, and a constant vertical acceleration component, a_y, the magnitude of which is equal to the magnitude of the acceleration due to gravity, g. ☞

9 *Uniform motion equations* may be applied to the horizontal motion of a projectile, giving

$$s_x = u_x t \quad \text{(Eqn 17)}$$

$$v_x = u_x = \text{constant} \quad \text{(Eqn 16)}$$

and $$a_x = 0 \quad \text{(Eqn 15)}$$

where the displacement, s_x, is taken to be zero when $t = 0$.

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10 *Uniform acceleration equations* may be applied to the vertical motion of a projectile, giving

$$v_y = u_y - gt \qquad \text{(Eqn 19)}$$

$$s_y = u_y t - \tfrac{1}{2} g t^2 \qquad \text{(Eqn 20)}$$

$$v_y^2 = u_y^2 - 2 g s_y \qquad \text{(Eqn 21)}$$

Where the *y*-axis has been assumed to point vertically upwards and the displacement, s_y, is taken to be zero when $t = 0$.

11 A projectile moving near the Earth, under the influence of gravity alone, follows a *parabolic trajectory* and achieves maximum horizontal *range* when launched at 45° to the horizontal.

12 The strategy used to solve projectile problems is to resolve the initial velocity into its horizontal and vertical components and subsequently to treat each component as an example of linear motion.

13 Similar principles and methods apply whenever a particle is subject to constant acceleration in a direction other than its direction of motion, or opposite to this direction.

14 Vector methods of the sort developed to treat two-dimensional projectile problems may be readily extended to deal with *three-dimensional* problems

5.2 Achievements

Having completed this module, you should be able to:

A1 Define the terms that are emboldened and flagged in the margins of the module.

A2 Describe in quantitative terms the magnitude and direction of two-dimensional vector quantities such as displacement, velocity and acceleration given their horizontal and vertical components.

A3 Draw and interpret graphical representations of vector quantities such as the position, displacement and velocity of a projectile.

A4 Recall and apply the equations of motion that describe the behaviour of projectiles. (Equations 15–21 for motion under gravity, and Equations 22–24 more generally.)

A5 Solve problems concerning the range, time of flight, maximum height, velocity and displacement of projectiles.

A6 Solve general problems involving two-dimensional motion with uniform acceleration.

Study comment You may now wish to take the *Exit test* for this module which tests these Achievements. If you prefer to study the module further before taking this test then return to the *Module contents* to review some of the topics.

5.3 Exit test

Study comment Having completed this module, you should be able to answer the following questions, each of which tests one or more of the *Achievements*.

Question E1

(*A3*) Sketch a Cartesian coordinate system, showing the origin and the point with position vector $\boldsymbol{r} = (1\,\text{m}, 2\,\text{m})$. What is the distance of this point from the origin?

Question E2

($A2$ and $A3$) A ball is at point A in Figure 11 at a time t_1 = 1.4 s and at point B at a time t_2 = 1.8 s. Find the displacement from A to B and hence find the average velocity vector over the time interval from t = 1.4 s to t = 1.8 s.

Question E3

($A3$, $A4$ and $A5$) The opening to a basketball net is 3.05 m above the ground. A player's feet are 6 m away from a mark on the floor immediately below the net. The player throws the basketball from a height of 2 m and at an angle of 60° above the horizontal. With what speed should the player throw the ball in order to get it into the net? (Assume g = 9.81 m s^{-2}, and that both the ball and net can be treated as point objects.)

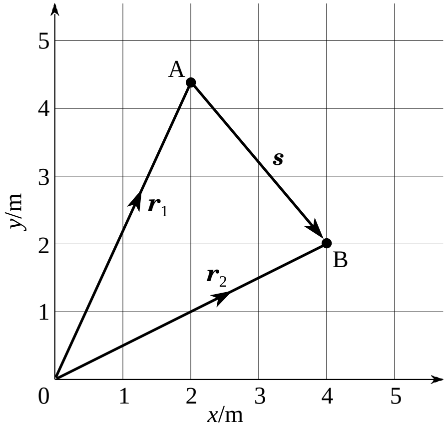

Figure 11 See Question E2.

Question E4

(*A4* and *A5*) A rifleman holds his rifle with its barrel horizontal at a height of 1.5 m above the ground. He is aiming at the centre of a target which is 100 m away and also at a height of 1.5 m. Assuming the muzzle speed of the bullet to be $500\,\mathrm{m\,s^{-1}}$, and that the barrel is lined up with the target, find the time taken for the bullet to reach the target and the vertical distance by which the bullet will miss the centre of the target. Take $g = 9.81\,\mathrm{m\,s^{-2}}$.

Question E5

(*A4* and *A5*) A stone is thrown horizontally from the top of a vertical cliff with a velocity of $15\,\mathrm{m\,s^{-1}}$. It is observed to reach the sea at a point 45 m from the foot of the cliff. Find the time of flight of the stone and the height of the cliff.

Question E6

(*A3*, *A4* and *A5*) A gun fires shells from ground level with a muzzle speed of $300\,\mathrm{m\,s^{-1}}$. In a test, shells are fired at (a) 30°, (b) 45° and (c) 60° to the horizontal. Assuming that $g = 9.81\,\mathrm{ms^{-2}}$ find the range and the maximum height reached for each angle of firing. Using centimetre-squared graph paper, sketch a scale drawing of each trajectory using the same set of axes.

Question E7

($A4$ and $A5$) Look back at the information given in Question T7

> In the 1976 Olympic Games the shot-putting event was won with a throw of 21.32 m. Assuming that the shot was thrown from ground level at the optimum angle (45°), calculate the speed at which it was projected, making use of the formula for range. (Assume $g = 9.81\,\mathrm{m\,s^{-2}}$.)

and recalculate the speed needed on the assumption that the shot was launched at 45°, but from a height of 2 m rather than from ground level.

Question E8

($A4$ and $A6$) An ice puck is travelling along the y-axis of the rink at $2\,\mathrm{m\,s^{-1}}$. At a particular instant, when its position coordinates are $(x, y) = (0\,\mathrm{m}, 5\,\mathrm{m})$, it begins to accelerate parallel to the x-axis with $a_x = 2\,\mathrm{m\,s^{-2}}$. Find its new position and velocity vectors after 5 s, and express each of these in both component and magnitude/direction notation.

Study comment This is the final *Exit test* question. When you have completed the *Exit test* go back to Subsection 1.2 and try the *Fast track questions* if you have not already done so.

If you have completed **both** the *Fast track questions* and the *Exit test*, then you have finished the module and may leave it here.